Beltrami never ceased to meditate on the non-Euclidean geometry even when concentrating all his powers to the study of natural phenomena. A proof of this is his discovery that the general equation of elasticity is bound to the Euclidean postulate. Moreover one of his gifted disciples observes "how he shows, in a certain passage, that he had turned his attention to the way in which physics would be able to profit from hypotheses of a diverse geometric nature of space, a difficult conception, more explicitly advanced by Clifford; nor was he ever able to lose from view those curved spaces, with which he had commenced so triumphantly."

Settled finally at Rome, member of the most celebrated scientific societies of the world, successor to Brioschi as President of the Accademia dei Lincei, Senator of the Realm of Italy, many times chosen by public vote to sit in the Council Superior of Public Instruction, acclaimed master by the entire body of scientists, happy with a devoted wife, yet from 1896 he was undermined by a mysterious malady, and died February 18th, 1900.

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ON THE PRIMITIVE GROUPS OF CLASS FOUR.

By DR. G. A. MILLER.

The class of a substitution group is the smallest number of elements in any one of its substitutions besides the identity. This definition seems to be due to Camille Jordan,* who made extensive investigations in regard to the primitive groups of small classes. It is well known that the alternating and the symmetric groups are of classes two and three respectively, and that these are the only primitive groups of these two classes. Moreover, Jordan proved that these are the only two classes for which the number of primitive groups is infinite,† and he investigated the problem of determining all the primitive groups whose class does not exceed 13, publishing only a brief outline of his work.‡

In his work on the *Theory of Substitutions*, pages 133 to 138, Netto gives an outline of a proof that there is no primitive group of class four and of degree greater than 8. As this theorem is of great importance in the theory of primitive groups, it appeared desirable to give a more complete proof, based upon some recent theorems. This proof will thus serve as another illustration of the application of these theorems, and it is hoped that it will tend to simplify one step towards the difficult subject of class of primitive substitution groups.

Let G be any primitive group of degree $n>8$ and of class four. The subgroup (H) generated by its substitutions of the form $ab.cd=s_1$, which may be supposed to be contained in G, must be invariant, since it includes all the con-

*Jordan, Liouville, vol. 16, 1871, page 383.
†Comptes Rendus, vol. 73, 1871, p. 853.
‡Loc. cit. vol. 75, p. 1757.

jugates of s_1. Hence it must be of degree n and transitive. It must be non-abelian since abelian transitive groups are regular. We shall first prove that H must include the regular four-group including s_1. This will prove that G is at least doubly transitive and that each one of its substitutions of degree four and order two is contained in a sub-group of order and degree four.

The sub-group H contains at least one substitution s_2 which is similar to s_1 and non-commutative with it. If s_1 and s_2 would have only one element in common their commutator $s_1^{-1}s_2^{-1}s_1s_2$ would be of degree three and G could not be of class four.* If they had three common elements they would generate a group of degree 5, which would be transitive since the intransitive groups of this degree are either of class 2 or of class 3. Every transitive group of degree p, p being any prime number, must include a cyclic substitution of degree p. It has been proved that such a substitution cannot occur in any primitive group whose class exceeds 3 except when the degree is one of the three numbers $p, p+1, p+2$.† Hence s_1 and s_2 must contain just two common elements; that is *any two non-commutative substitntions of G which are similar to s_1 have just two common elememts.*

The two substitutions s_1 and s_2 must therefore generate a dihedral rotation group of degree six whose order is either 6 or 8.‡ In the latter case s_1 is evidently included in a regular four-group. In the former case, the group generated by s_1 and s_2 is the intransitive group obtained by establishing a simple isomorphism between two symmetric groups of degree three. We proceed to prove that in this case s_1 must also be included in a regular four-group contained in H.

Only three of the substitutions of H which are similar to s_1 have been determined, viz: $s_1, s_2, s_1^{-1}s_2s_1$. If all the other similar substitutions were commutative with s_1 they would also have to be commutative with each of the other two given conjugates, since these three substitutions are transformed transitively by a sub-group which transforms all the rest among themselves. Hence H includes another substitution (s_3) which is not commutative with s_1, and therefore it has just two elements in common with s_1.

If the group generated by s_1, s_2, s_3 were transitive it would include a regular four-group containing s_1 since the degree of this group would be either six or seven and it may be assumed that it would not contain any substitution of order 5. As it would also be positive and of class four there are only two groups which require consideration, viz: $(+abcdef)_{24}$ and $(abcdefg)_{168}$.§ It remains to consider the cases when s_1, s_2, s_3 would generate one of the following intransitive groups: (1) A simple isomorphism between two symmetric groups of degree four, 2) A (4, 4) correspondence between these groups, and 3) A (1, 4) correspondence between the symmetric groups of degrees three and four respectively.

In the last two cases G would be at least doubly transitive and hence it

*Bochert, Mathematische Annalen, vol. 40, 1892, p. 159.

†Bulletin of the American Mathematical Society, vol. 4, 1898, p. 141.

‡Loc. cit., vol. 7, 1901, p. 424.

§American Jonrnal of Mathematics, vol. 21, p. 287.

would contain a substitution similar to s_1 which would permute its systems; that is, G would contain the regular four-group including s_1. In the first case G would contain an additional substitution (s_4) similar to s_1 and not commutative with s_1. If the group generated by s_1, s_2, s_3, s_4 were not a simple isomorphism between two symmetric groups of degree five it would clearly include a regular four-group containing a conjugate of s_1 and hence also such a group containing s_1. As this remark applies to all the following cases and as the substitutions which are similar to s_1 generate a transitive group, it follows that *in every primitive group of class four and degree* greater than 8 each substitution of type *ab.cd* is contained in the regular four-group. It may be observed that this applies also to the primitive groups of degrees 7 and 8 but not to those of degrees 5 and 6.

In what follows it may therefore be assumed that G is at least doubly transitive and that each of its substitutions similar to s_1 is contained in a regular four-group. It has already been observed that s_1, s_2 generate either the positive octic group of degree six or the intransitive group of degree and order six. In either case it may be assumed that s_1, s_2, s_3 generate $(+abcdef)_{24}$ since this is the only positive transitive group of degree six and class four which is generated by substitutions similar to s_1 and in which each of these substitutions is in a regular four-group. As G must contain some additional substitution (s_5) similar to s_1 which is not commutative with some one of the three conjugate substitutions of order 2 in $(+abcdef)_{24}$ and as the order of the group generated by s_1, s_2, s_3, s_5 must exceed 48, H must include a transitive group of degree seven or the alternating group of degree 6, the latter being the only positive group of degree six which includes $(+abcdef)_{24}$.

The only transitive group of degree seven and of class four is the well known simple group $(abcdefg)_{168}$. It follows from the theorem quoted above that this could not occur in a primitive group whose degree exceeds 9 unless this primitive group were either alternating or symmetric. It could not occur in a primitive group of degree 9 since all its 21 substitutions similar to s_1 are conjugate and hence each of these substitutions would be transformed into itself by $5.8=40$ substitutions of the primitive group of degree 9. This is clearly impossible since its order would not be divisible by 5. Hence there is no primitive group of class four and of degree greater than 8.

From what precedes and from the enumeration of the groups of degree 8* it follows that there are just six primitive groups of class four—two of each of the degrees 5 and 6 and one of each of the degrees 7 and 8. Their orders are 10, 20, 60, 120, 168, and 1344 respectively. The first two are the semi-metacyclic and the metacyclic groups† of degree 5. The third and fourth are, respectively, simply isomorphic with the alternating and the symmetric groups of degree six. They are the only instances of transitive groups of degree n and of orders $\frac{1}{2}(n-1)!$ and $(n-1)!$ and have been studied very fully in connection with the theory of equations of degree six. The first of these is known as the

*Cf. American Journal ol Mathematics, vol. 21, p. 287.

†Oeuvres de Lagrange, vol, 3, p. 339.

icosahedron rotation group and it is the smallest simple group of composite order. The fifth is very well known in the theory of elliptic modular functions and is the second smallest simple group of composite order. Kirkman remarks:* "Betti, Kronecker, Hermite, and myself have spent much time on this group." The last of these six primitive groups is the holomorph of the group of order 8 which includes no operator of order four.†

*Kirkman, Proceedings of the Manchester Literary and Philosophical Society, vol. 3, p. 65.
†American Journal of Mathematics, loc. cit.

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FACTORS OF A CERTAIN DETERMINANT OF OREER SIX.

By DR. L. E. DICKSON.

The following is an example of the so-called Group-Determinant:*

$$D\equiv\begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}$$

Note that the elements of the first three rows form two cyclic determinants of order three, and that the elements of the last three rows form the same two cyclic determinants. It follows readily that D has the factors $(\mathrm{I}+\alpha+\beta)\pm(\gamma+\delta+\varepsilon)$.

Upon adding to the first column all the remaining columns, we obtain an equal determinant having $\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon$ throughout the first column. Let D_1 be the determinant obtained by removing this factor, so that the elements in the first column of D_1 are all unity. Subtracting the first row from the remaining rows, we find that

$$D_1=\begin{vmatrix} \mathrm{I}-\alpha & \alpha-\beta & \delta-\gamma & \varepsilon-\delta & \gamma-\varepsilon \\ \beta-\alpha & \mathrm{I}-\beta & \varepsilon-\gamma & \gamma-\delta & \delta-\varepsilon \\ \delta-\alpha & \varepsilon-\beta & \mathrm{I}-\gamma & \alpha-\delta & \beta-\varepsilon \\ \varepsilon-\alpha & \gamma-\beta & \beta-\gamma & \mathrm{I}-\delta & \alpha-\varepsilon \\ \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & \mathrm{I}-\varepsilon \end{vmatrix}$$

From the first row, subtract the third, fourth, and fifth rows, and to the first row add the second row. In the resulting determinant, the elements of the first row are all divisible by $\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon$. Hence

*Its matrix forms the body of a left-hand multiplication-table for the symmetric group on three letters, where

$$\mathrm{I}=\text{identity},\ \alpha=(123),\ \beta=(132),\ \gamma=(12),\ \delta=(13),\ \varepsilon=(23).$$

Compare Weber, *Algebra*, 2nd Edition, Vol, II, page 124.